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Note

A Counterexample to a Discrete Korovkin Theorem*

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In this note a counterexample to a discrete Korovkin theorem, which has been proven by G. A. Anastassiou in this journal, is given; then, modifying Anastassiou's argument, I prove another discrete Korovkin theorem.

In [1] Anastassiou proved the following discrete Korovkin theorem.

THEOREM. *Let $X=\{x_1, ..., x_j, ...\}$ be a countable set. Consider $B(X)$, the space of real valued bounded functions on X with the supremum norm $\|\cdot\|_\infty$, and a sequence of positive linear operators L_n: $B(X)\to B(X)$ such that $L_n(1, x_j)=1$ for all j. Suppose that, for some $\{f_1, ..., f_k\}\subset B(X)$,*

$$\lim_{n\to\infty} L_n(f_i, x_j)=f_i(x_j) \qquad \textit{for all } i \textit{ and } j. \tag{1}$$

In order for $L_n(f, x_j)\to f(x_j)$ for all $f\in B(X)$ and all x_j, it is enough to assume for each j that there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x)-f_i(x_j))\geqslant 0 \qquad \textit{for all} \quad x\in X$$

and

$$\sum_{i=1}^{k} \beta_i(f_i(x)-f_i(x_j))>0 \qquad \textit{for all} \quad x\in X\setminus\{x_j\}.$$

* *Editorial footnote.* As noted above, this paper was received by the Editorial Office in 1986. Unfortunately it was not given a complete review until June 1990. In the meantime a paper by Dr. G. A. Anastassiou correcting his original paper (*J. Approx. Theory* **45** (1985), 383–388) in an essentially identical manner to that found in this paper has appeared (cf. *J. Approx. Theory* **61** (1990), 384–386). The editors deeply regret the delay in handling of this paper.

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In this note, constructing a simple counterexample, I remark that the theorem is not valid.

COUNTEREXAMPLE. Since the set of all rational numbers x with $0 \leqslant x \leqslant 1$ is countable, let us denote it by $X = \{x_1, ..., x_j, ...\}$. For each $n \geqslant 1$, define the positive linear operator L_n: $B(X) \to B(X)$ by

$$L_n(f, x) = \begin{cases} f(i/n) & \text{if } \ (i-1)/n < x \leqslant i/n \text{ with } i \geqslant 2, \\ f(1/n) & \text{if } \ 0 \leqslant x \leqslant 1/n. \end{cases}$$

Clearly, $L_n(1, x_j) = 1$ for all j; and if f is the function on X defined by $f(0) = 1$ and $f(x) = 0$ for all $x \neq 0$, then we have

$$\lim_{n \to \infty} L_n(f, x) = 0 \qquad \text{for all} \quad x \in X.$$

So $L_n(f, 0) \nrightarrow f(0)$. On the other hand, if f_1, f_2, and f_3 are the functions on X defined by $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$, respectively, then $\{f_1, f_2, f_3\}$ satisfies (2) (see [1]). Further (1) follows immediately. Consequently, the theorem does not hold.

Next I remark that Anastassiou's argument may be modified to yield the following

PROPOSITION. *Let X, $B(X)$, and L_n be as in the theorem. Let $x_j \in X$ be fixed arbitrarily. Suppose that, for some $\{f_1, ..., f_k\} \subset B(X)$,*

$$\lim_{n \to \infty} L_n(f_i, x_j) = f_i(x_j) \qquad \text{for all } i. \tag{3}$$

In order for $L_n(f, x_j) \to f(x_j)$ for all $f \in B(X)$, it is enough to assume that there are real constants $\beta_1, ..., \beta_k$ and a positive constant $\delta > 0$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_j)) \geqslant \delta \qquad \text{for all} \quad x \in X \backslash \{x_j\}. \tag{4}$$

Sketch of Proof. We choose a positive finite measure μ on X with $\mu(\{x\}) > 0$ for all $x \in X$. Then $B(X) = L_\infty(X, \mu)$, and each positive linear functional $L_n(\cdot, x_j)$ on $B(X)$ is bounded on $L_\infty(X, \mu)$. Therefore there exists a positive finitely additive measure $g_n \in L_\infty^*(X, \mu)$ such that

$$L_n(f, x_j) = \langle f, g_n \rangle = \int_X f \, dg_n \qquad \text{for all} \quad f \in B(X).$$

By $L_n(1, x_j) = 1$, we have $\int_X 1 \, dg_n = 1$. Thus, as in the proof of Proposition 5 in [1], we see that if there exists an $f \in B(X)$ with

$\lim_{n\to\infty} L_n(f, x_j) \neq f(x_j)$ then there cannot be constants $\delta > 0$ and $\alpha_1, ..., \alpha_k$ such that

$$\sum_{i=1}^{k} \alpha_i(f_i(x) - f_i(x_j)) \geqslant \delta \qquad \text{for all} \quad x \in X \setminus \{x_j\}.$$

This contradicts (4), and the proof is completed.

Reference

1. G. A. Anastassiou, A discrete Korovkin theorem, *J. Approx. Theory* **45** (1985), 383–388.